MATH4210: Financial Mathematics Tutorial 2

Xiangying Pang

The Chinese University of Hong Kong

xypang@math.cuhk.edu.hk

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Exercise

Let $(Y_j)_{j\in\mathbb{N}}$ be a sequence of i.i.d. random variables. For any $j\in\mathbb{N}, \mathbb{P}(Y_j=\pm 1)=\frac{1}{2}$. Define for $n\in\mathbb{N}, X_n=\sum_{j=1}^{n} Y_j$. Show that $(X_n)_{n\in\mathbb{N}}$ is a martingale.

Definition for discrete-time martingale:

$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \cdots$

Let $\mathcal{F}_n$ be a filtration, that is, an increasing sequence of $\sigma-$fields. A sequence X_n is said to be adapted to $\mathcal{F}_n$ if $X_n \in \mathcal{F}_n$. If X_n is sequence with (1) $\mathbb{E}|X_n|<\infty$, (2) X_n is adapted to $\mathcal{F}_n$, (3) $\mathbb{E}(X_{n+1}|\mathcal{F}_n)=X_n$ for all n . then X_n is said to be a **martingale with respect to $\mathcal{F}_n$**.

X_n is $\mathcal{F}_n$-measurable

Solution:

1. Fix $n\in\mathbb{N}$.

$$\mathbb{E}(|X_n|)=\mathbb{E}(|\sum_{j=1}^{n} Y_j|)$$

$$\leq \sum_{j=1}^{n}\mathbb{E}(|Y_j|)$$

(1) ✓

$$=n(1*\frac{1}{2}+|-1|*\frac{1}{2})=n<\infty$$

$(X_n) \xrightarrow{\text{def}} \mathcal{F}_n$

2. Fix $n \in \mathbb{N}$, denote $\mathcal{F}_n = \sigma(X_0, ..., X_n)$. $\rightarrow X_n \in \mathcal{F}_n$ (2) ✓

$$
\begin{aligned}
\mathbb{E}(X_{n+1}|\mathcal{F}_n) &= \mathbb{E}(X_n + Y_{n+1}|\mathcal{F}_n) \\
&\overset{\text{linear}}{=} \mathbb{E}(X_n|\mathcal{F}_n) + \mathbb{E}(Y_{n+1}|\underbrace{\mathcal{F}_n}_{= \sigma(X_0, \dots, X_n) = \sigma\left(\sum_{j=1}^{n} Y_j\right)}) \\
&\overset{\text{ind}}{=} X_n + \underbrace{\mathbb{E}(Y_{n+1})}_{=0} \\
&= X_n
\end{aligned}
$$

(3) ✓

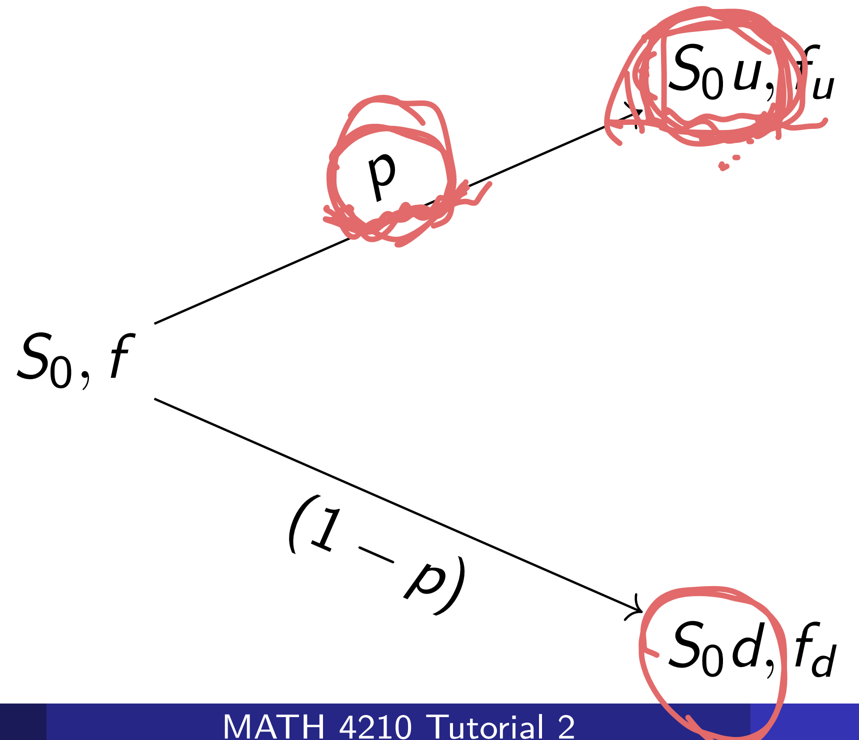
By 1 and 2, $(X_n)_{n\in\mathbb{N}}$ is a martingale.

Remark

It still works when $\mathbb{P}(Y_j = 2) = \frac{1}{3}$ and $\mathbb{P}(Y_j = -1) = \frac{2}{3}$. More generally, if $\mathbb{E}(Y_j) = 0$, $(X_n)_{n\in\mathbb{N}}$ will still be a martingale (Exercise!).

Binomial Tree Models

Consider the following problem.
Suppose we target on a stock whose price is S_t, $t = 0, 1$. t represents the time. Now we are at time $t = 0$, and we can observe the stock price S_0. At time $t = 1$, it has two possibilities of moving to either $S_1 = S_0u$ or $S_1 = S_0d$ for some $u > 1$ and $0 < d < 1$ with probability p and $1 - p$ respectively. Suppose we have an option whose underlying asset is the stock maturing at $t = 1$. If the stock price becomes $S_0u(S_0d)$, the value of the option is $f_u(f_d)$. How can we find the option price f at time $t = 0$?



Binomial tree Models

$n=1$

$$S_{t_1} < \begin{matrix} S_0 u \\ S_0 d \end{matrix}$$

cash ϕ

$$G(S_{t_0}, S_{t_1}) < \begin{cases} f_u = V_{t_1}^{x,\phi} = (x-\phi S_0)(1+r\Delta t) + \phi S_0 u \\ f_d = V_{t_1}^{x,\phi} = (x-\phi S_0)(1+r\Delta t) + \phi S_0 d \end{cases}$$

$$\Rightarrow \begin{cases} x = (1+r\Delta t)^{-1}[q f_u + (1-q) f_d] \\ \phi = \dfrac{f_u - f_d}{S_0 u - S_0 d} \\ q = \dfrac{1+r\Delta t - d}{u-d} \end{cases}$$

$E^{Q}(f_{t_1})$

$$G(S_{t_1}) = (S_{t_1}-K)^+ < \begin{matrix} (S_0 u - K)^+ \\ (S_0 d - K)^+ \end{matrix} \Rightarrow (x, \phi)$$

Proposition (Propotion 1.1 in Slide 3)

Let $G(S_{t_0}, S_{t_1}, \cdots, S_{t_n})$ be the payoff of a derivative option at maturity t_n. Assume that there is some (x, ϕ) such that

principle strategy

$G(S_{t_0}, S_{t_1}, \cdots, S_{t_n}) = V_{t_n}^{x,\phi}$, a.s.then the price of the option should be x.

(x, ϕ)

Remark

If the option is not well priced, there will be arbitrage opportunities for option buyers or sellers.

Binomial Tree Models

If $q \in (0, 1)$, it defines a probability measure $\mathbb{Q}$ (totally unrelated to p) that,

$$\begin{cases} \mathbb{Q}[S_{t_1} = S_0 u] & = \mathbb{Q}[f_{t_1} = f_u] = q \\ \mathbb{Q}[S_{t_1} = S_0 d] & = \mathbb{Q}[f_{t_1} = f_d] = 1 - q. \end{cases}$$

Then, we have

$$f = (1 + r\Delta t)^{-1} \mathbb{E}^{\mathbb{Q}}[f_{t_1}] := \sum_{f_i} f_i \times \mathbb{Q}[f_{t_1} = f_i]$$

Note that we do Not necessarily have

$$p := \mathbb{P}(S_{t_1} = S_0 u) = q.$$

If this is the case, we call it in the risk neutral world.

1\$ (1+rt)\$

0 t

Remark

- we call r is a discrete compound rate if we only compound the interest on each time periods. For example, starting at $t = 0$, we compound the interest at $t = \Delta t, 2\Delta t, \ldots$ and the interest values $(1 + r\Delta t)^1, (1 + r\Delta t)^2, \ldots$
- we call r a continuous compound rate if we continuously compound the interest on all $t \geq 0$. At any $t \geq 0$, the interest values e^{rt}.

1\$ $(1+r\Delta t)$, $(1+r\Delta t)^2$ $\to (1+r\Delta t)^n$ + $n\to\infty$

0 Δt $2\Delta t$ $t = n\Delta t$

$= \lim_{n\to\infty} (1+\frac{rt}{n})^n$

$= \lim_{n\to\infty} (1+\frac{1}{\frac{n}{rt}})^{\frac{n}{rt}\cdot rt}$

$= \lim_{y=\frac{n}{rt}\to\infty} (1+\frac{1}{y})^{y\cdot rt} = e^{rt}$

$(1+rt)^k \to e^{rt\cdot k}$

discrete continuous

(a) $S_0 u = 20 \times 1.2 = 24$; $f_u = (S_0 u - 20)^+ = 4$

$S_0 = 20$

$S_0 d = 20 \times 0.67 = 13.4$; $f_d = (S_0 d - 20)^+ = 0$

$r = 0.1$

$x = (1 + r\Delta t)^{-1}[q f_u + (1-q) f_d] = 2.9$

$q = \frac{1 + r\Delta t - d}{u - d} = \frac{1.1 - 0.67}{1.2 - 0.67} = 0.81$

Question

Given the current price of the underlying stock, $S_0 = 20$ $(= S_{t_0})$*. The stock price goes up and down by* $u = 1.2$ *and* $d = 0.67$*, respectively. The one period risk-free interest rate is* 10%.

a) Price a one period European call option with exercise price $K = 20$*. Consider the discrete compound case.*

$t_1 = 1$ $(S_{t_1} - K)^+$

b) Price a one period European call option with exercise price $K = 20$*. Consider the continuous compound case.*

(b)

$x = e^{-r\Delta t}[q f_u + (1-q) f_d] = e^{-0.1}[q f_u + (1-q) f_d] = 2.97$

$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.1} - 0.67}{1.2 - 0.67} = 0.82$

$e^x \to 1 + x + o(x^2)$

$(1 + r\Delta t)^k \to e^{r\Delta t \cdot k}$